Boundary-Layer Behavior on Continuous Solid Surfaces:

II. The Boundary Layer on a Continuous Flat Surface

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The laminar boundary-layer behavior on a moving continuous flat surface is investigated by two methods. One method involves the numerical solution of the boundary-layer equations. The other is an integral method, based on an assumed velocity profile that satisfies the appropriate boundary conditions. Good agreement is obtained between the results of these two methods of solution. The turbulent boundary-layer behavior on a moving continuous flat surface is investigated by the integral method only. Equations for the boundary-layer thickness, displacement thickness, momentum thickness, and skin friction are presented for both laminar and turbulent boundary layers. Comparison is made with the boundary-layer behavior over a flat plate of finite length.

In a previous paper (2) the boundary-layer behavior on continuous surfaces was examined and compared with the boundary-layer behavior on surfaces of finite length. In the same paper the differential equations of motion and boundary conditions that determine the fluid behavior were examined, and the momentum integral equations applicable to continuous surfaces were derived. The results of that study indicate that the known solutions for the boundary layer on surfaces of finite length are not applicable to the boundary layer on continuous surfaces.

In this paper the boundary-layer equations will be solved to determine the behavior of laminar and turbulent boundary layers on a continuous flat surface. The laminar boundary layer will be investigated by two methods, one involving the numerical solution, with appropriate boundary conditions, of the differential boundary-layer equations and the other involving the previously derived (2) momentum integral equation of boundary-layer theory. In this solution the assumed velocity profile is a polynomial of fourth degree that satisfies the appropriate boundary conditions. The turbulent boundary layer will be investigated by the momentum integral method only. The characteristic boundary-layer parameters will be derived and compared with the corresponding parameters for a flat plate of finite length.

THE LAMINAR BOUNDARY LAYER ON A CONTINUOUS FLAT SURFACE

Exact Solution

Consider steady, two-dimensional, incompressible flow on a continuous flat surface moving with a constant velocity in a fluid medium at rest, as shown in Figure 1. The adopted frame of axes is stationary with its center point (at $x = 0$) located at the slot. The positive x axis extends parallel to the surface and in the direction of its motion. For the upper part of the surface the positive y axis originates at the surface ($y = 0$) and extends upwards. The boundary-layer equations are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

The appropriate boundary conditions are

$$u = U_f, \quad v = 0 \text{ at } y = 0$$

$$u \to 0 \qquad \text{at } y \to \infty$$

A stream function of the form

$$\psi = \sqrt{\nu x \; U_f} \, f(\eta) \qquad (3)$$

where $f(\eta)$ denotes the dimensionless stream function and

$$\eta = y \sqrt{\frac{U_f}{\nu x}} \qquad (4)$$

is introduced to satisfy the continuity equation [Equation (2)]. The velocity components now become

$$u = U_f f'; v = \frac{1}{2} \sqrt{\frac{\nu U_f}{x}} (\eta f' - f) \qquad (5)$$

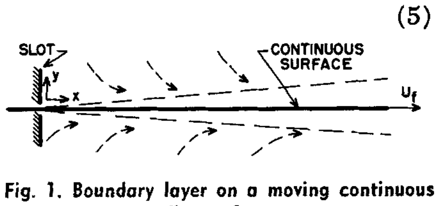

Fig. 1. Boundary layer on a moving continuous flat surface.

where the prime denotes differentiation with respect to η. Substitution in Equation (1) and simplification leads to the following differential equation:

$$ff'' + 2f''' = 0 \qquad (6)$$

The boundary conditions now become

$$f = 0, \; f' = 1.0 \text{ at } \eta = 0$$

$$f' \to 0 \text{ at } \eta \to \infty$$

Equation (6) is the well-known Blasius equation for the flat plate of finite length (3). However the boundary conditions that must be satisfied in solving Equation (6) for a continuous flat surface are not the same as for the flat plate of finite length. Hence, as already inferred (2), the solution to Equation (6) will be different from the Blasius solution.

Equation (6) was solved by the fourth-order Runge-Kutta method on an IBM card-programmed calculator, and the results to five decimal figures are given in Table 1. The programing of the solution of Equation (6) was checked by solving it to satisfy the boundary conditions appropriate to the flat plate of finite length. Howarth's values of f, f', and f'', as given in reference 3 for this case, were duplicated. The accuracy of the solution given in Table 1 was checked by recomputing it with interval sizes of ½ and twofold that used in obtaining Table 1. The solution given in Table 1 was actually carried to $\eta = 23$.

The variation of the axial velocity component $u/U_f = f'(\eta)$ is plotted in Figure 2. The transverse velocity component in the boundary layer, given by Equation (5), is plotted in Figure 3. Note that at the outer edge of the boundary layer the transverse velocity component differs from zero; that is for $\eta \to \infty$

$$v = -0.808 \; U_f \sqrt{\frac{\nu}{U_f x}}$$

The negative sign signifies influx of fluid across the limit of the boundary layer.

The primary characteristic boundary-layer parameters for the moving

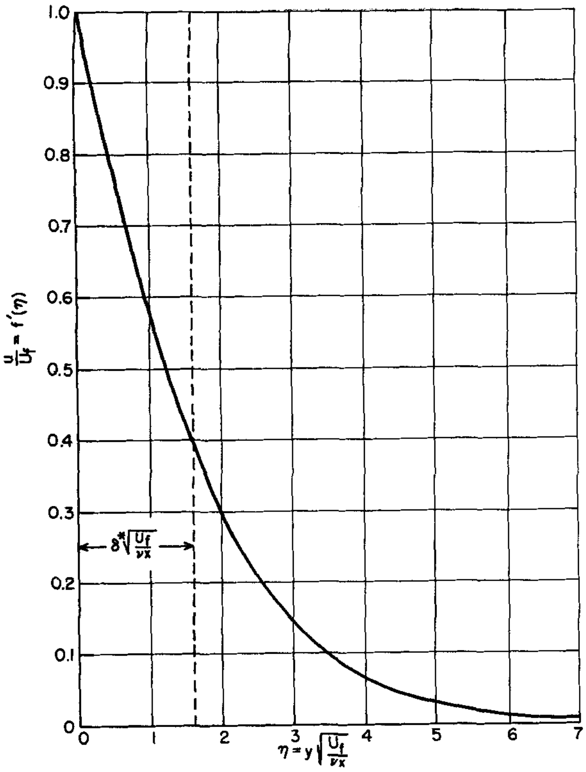

Fig. 2. Velocity distribution in the boundary layer along a continuous flat surface.

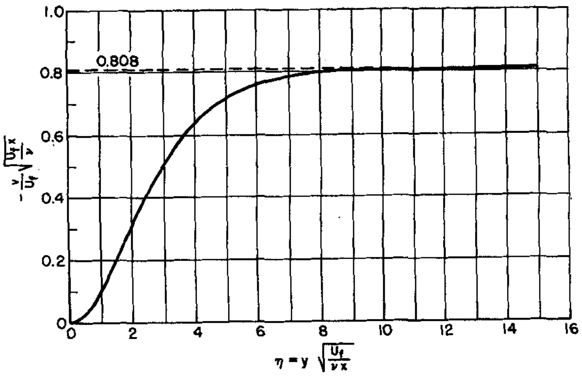

Fig. 3. The transverse velocity component in the boundary layer along a continuous flat surface.

continuous surface can now be computed readily. These are

$$\theta = \frac{1}{U_f^2} \int_{y=0}^{y=\infty} u^2 dy \qquad (7a)$$

or

$$\theta = \sqrt{\frac{\nu x}{U_f}} \int_{\eta=0}^{\eta=\infty} f'^2 dy \qquad (7b)$$

the momentum thickness*, and

$$\int_{\eta=0}^{\eta=\infty} f'^2 d\eta = 0.888.$$

$$\delta^* = \frac{1}{U_f} \int_{y=0}^{y=\infty} u dy \qquad (8a)$$

or

$$\delta^* = f(\eta_1) \sqrt{\frac{\nu x}{U_f}} \qquad (8b)$$

the displacement thickness, which is the distance by which the external stationary field is displaced outwards as a consequence of the increase in velocity in the boundary layer. Here η_1 denotes a point outside the boundary.

If the boundary-layer thickness is defined as that distance from the moving surface at which u = 0.01 U_f, then, as seen from Table 1, $\eta = 6.37$.

From these results the secondary boundary-layer parameters are obtained as follows:

$$\tau_f = -\frac{\mu}{g_c}\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (9a)$$

or

$$\tau_f = -\frac{\mu}{g_c} U_f \sqrt{\frac{U_f}{\nu x}} f''(0) \qquad (9b)$$

the local shearing stress or skin friction at the surface

$$C'_f = \frac{g_c \tau_f}{\frac{1}{2}\rho U_f^2} = 2\left(\frac{d\theta}{dx}\right) \qquad (10)$$

the local drag coefficient

$$D = 2b \int_{x=0}^{x=L} \tau_f dx = \frac{2b\rho}{g_c} U_f^2 \theta_L \qquad (11)$$

the drag on both sides of the continuous surface

$$C_f = \frac{g_c D}{\frac{1}{2}\rho A U_f^2} = 2\frac{\theta_L}{L} \qquad (12)$$

the total drag coefficient

$$q = 2b \int_{y=0}^{y=\infty} u dy = 2bU_f \delta^*_L \qquad (13)$$

and the pumping action, or total volume of fluid entrained, by the moving continuous surface.

The pertinent results are given in Table 2.

Approximate Solution

The complete calculation of the boundary layer for a given continuous surface by means of the differential equations of motion without recourse to a computer is cumbersome and *time-consuming.* To facilitate these computations approximate methods are needed which would give useful results even if their accuracy is somewhat limited. One such approximate method makes use of the integral momentum equation for the boundary layer. The momentum equation for the boundary layer on moving continuous surfaces has already been derived (*2*). For a moving continuous flat surface the momentum equation is

$$\frac{d}{dx}(U_f^2\theta) = \frac{g_c \tau_f}{\rho} \qquad (14)$$

The momentum thickness is defined by Equation (7). The physical meaning of Equation (14) is that the shear stress at the surface is equal to the gain of momentum in the boundary layer.

Assuming that the velocity profiles in the boundary layer are similar one can put

$$\frac{u}{U_f} = f\left(\frac{y}{\delta}\right) = f(\hat{\eta}) \qquad (15)$$

where $\hat{\eta} = y/\delta$ is the dimensionless distance from the wall referred to the boundary-layer thickness.

The choice of the form of the function $f(\hat{\eta})$ is limited by a number of conditions that must be satisfied. For this purpose a polynomial of the fourth degree for the velocity function in terms of the dimensionless distance from the moving surface, $\hat{\eta} = y/\delta$, was assumed:

$$\frac{u}{U_f} = f(\hat{\eta}) = a + b\hat{\eta} + c\hat{\eta}^2 + d\hat{\eta}^3 + e\hat{\eta}^4 \qquad (16)$$

The free constants *a*, *b*, *c*, *d*, and *e* were evaluated by forcing Equation (16) to satisfy the following boundary conditions:

$u = U_f$ at $y = 0$ or $f = 1.0$ at $\hat{\eta} = 0$

$u = 0$ at $y = \delta$ or $f = 0$ at $\hat{\eta} = 1.0$

$\frac{\partial u}{\partial y} = 0$ at $y = \delta$ or $\check{f'} = 0$ at $\eta = 1.0$

$\frac{\partial^2 u}{\partial y^2} = 0$ at $y = \delta$ or $f'' = 0$ at $\hat{\eta} = 1.0$

$\frac{\partial^2 u}{\partial y^2} = 0$ at $y = 0$ or $f'' = 0$ at $\hat{\eta} = 0$

Thus Equation (16) becomes

* This required further numerical computation of the integral

Table 1. The Function $f(\eta)$ for the Boundary Layer on a Continuous Plane Solid Surface

$\eta = y\sqrt{\frac{U_f}{\nu x}}$	f	$f' = \frac{u}{U_f}$	$-f''$	f'''
0	0	1.00000	0.44375	0
0.01	0.00998	0.99556	0.44374	0.00221
0.02	0.01991	0.99113	0.44371	0.00442
0.03	0.02980	0.98669	0.44365	0.00661
0.04	0.03964	0.98225	0.44357	0.00879
0.05	0.04945	0.97782	0.44347	0.01096
0.06	0.05920	0.97338	0.44335	0.01312
0.07	0.06891	0.96895	0.44321	0.01527
0.08	0.07858	0.96452	0.44305	0.01741
0.09	0.08820	0.96009	0.44286	0.01953
0.10	0.09778	0.95566	0.44266	0.02164
0.15	0.14501	0.93356	0.44132	0.03200
0.20	0.19114	0.91154	0.43946	0.04200
0.25	0.23617	0.88962	0.43712	0.05162
0.30	0.28010	0.86783	0.43431	0.06083
0.35	0.32295	0.84620	0.43105	0.06960
0.40	0.36473	0.82474	0.42736	0.07793
0.45	0.40543	0.80347	0.42326	0.08580
0.50	0.44508	0.78242	0.41878	0.09320
0.55	0.48368	0.76160	0.41395	0.10011
0.60	0.52124	0.74103	0.40878	0.10654
0.65	0.55778	0.72072	0.40330	0.11248
0.70	0.59332	0.70070	0.39754	0.11793
0.75	0.62786	0.68097	0.39152	0.12291
0.80	0.66142	0.66155	0.38526	0.12741
0.85	0.69402	0.64245	0.37879	0.13144
0.90	0.72567	0.62368	0.37212	0.13502
0.95	0.75639	0.60524	0.36529	0.13815
1.00	0.78620	0.58715	0.35831	0.14085
1.20	0.89666	0.51836	0.32936	0.14766
1.40	0.99394	0.45546	0.29962	0.14890
1.60	1.07924	0.39850	0.27009	0.14575
1.80	1.15373	0.34736	0.24154	0.13934
2.00	1.21855	0.30178	0.21451	0.13069
2.20	1.27479	0.26143	0.18935	0.12069
2.40	1.32344	0.22590	0.16627	0.11003
2.60	1.36544	0.19478	0.14535	0.09923
2.80	1.40162	0.16762	0.12656	0.08870
3.00	1.43273	0.14401	0.10984	0.07868
3.20	1.45943	0.12356	0.09504	0.06936
3.40	1.48233	0.10588	0.08204	0.06081
3.60	1.50195	0.09063	0.07067	0.05307
3.80	1.51873	0.07751	0.06076	0.04614
4.00	1.53307	0.06624	0.05216	0.03998
4.20	1.54533	0.05657	0.04472	0.03455
4.40	1.55579	0.04828	0.03830	0.02979
4.60	1.56472	0.04119	0.03276	0.02563
4.80	1.57234	0.03513	0.02801	0.02202
5.00	1.57883	0.02995	0.02392	0.01889
5.20	1.58437	0.02552	0.02042	0.01618
5.40	1.58908	0.02174	0.01743	0.01385
5.60	1.59310	0.01852	0.01486	0.01184
5.80	1.59652	0.01577	0.01267	0.01012
6.00	1.59944	0.01343	0.01080	0.00864

$\eta = y\sqrt{\frac{U_f}{\nu x}}$	f	$f' = \frac{u}{U_f}$	$-f''$	f'''
6.20	1.60192	0.01143	0.00920	0.00737
6.40	1.60403	0.00973	0.00784	0.00629
6.60	1.60583	0.00828	0.00668	0.00536
6.80	1.60736	0.00705	0.00569	0.00457
7.00	1.60866	0.00600	0.00484	0.00389
7.20	1.60977	0.00511	0.00412	0.00332
7.40	1.61071	0.00434	0.00351	0.00283
7.60	1.61151	0.00370	0.00299	0.00241
7.80	1.61219	0.00314	0.00254	0.00205
8.00	1.61278	0.00267	0.00216	0.00174
8.20	1.61327	0.00228	0.00184	0.00149
8.40	1.61369	0.00193	0.00157	0.00126
8.60	1.61405	0.00165	0.00133	0.00108
8.80	1.61435	0.00140	0.00113	0.00092
9.00	1.61461	0.00119	0.00097	0.00078
9.20	1.61483	0.00101	0.00082	0.00066
9.40	1.61502	0.00086	0.00070	0.00056
9.60	1.61517	0.00073	0.00059	0.00048
9.80	1.61531	0.00062	0.00051	0.00041
10.00	1.61542	0.00053	0.00043	0.00035
10.20	1.61552	0.00045	0.00037	0.00030
10.40	1.61560	0.00038	0.00031	0.00025
10.60	1.61567	0.00032	0.00027	0.00021
10.80	1.61573	0.00027	0.00023	0.00018
11.00	1.61578	0.00023	0:00019	0.00016
11.20	1.61583	0.00020	0.00016	0.00013
11.40	1.61586	0.00017	0.00014	0.00011
11.60	1.61589	0.00014	0.00012	0.00010
11.80	1.61592	0.00012	0.00010	0.00008
12.00	1.61594	0.00010	0.00009	0.00007
12.20	1.61596	0.00008	0.00007	0.00006
12.40	1.61598	0.00007	0.00006	0.00005
12.60	1.61599	0.00006	0.00005	0.00004
12.80	1.61600	0.00005	0.00004	0.00004
13.00	1.61601	0.00004	0.00004	0.00003
13.20	1.61602	0.00003	0.00003	0.00003
13.40	1.61602	0.00003	0.00003	0.00002
13.60	1.61603	0.00002	0.00002	0.00002
13.80	1.61603	0.00002	0.00002	0.00002
14.00	1.61604	0.00002	0.00002	0.00001
14.20	1.61604	0.00001	0.00001	0.00001
14.40	1.61604	0.00001	0.00001	0.00001
14.60	1.61604	0.00001	0.00001	0.00001
14.80	1.61604	0.00001	0.00001	0.00001
15.00	1.61605	0.00000	0.00001	0.00001
15.20	1.61605	0.00000	0.00001	0.00001
15.40	1.61605	0.00000	0.00001	0.00000
15.60	1.61605	0.00000	0.00000	0.00000

$$\frac{u}{U_f} = f(\hat{\eta}) = 1 - 2\hat{\eta} + 2\hat{\eta}^3 - \hat{\eta}^4 \tag{17}$$

The velocity profile given by Equation (17) is plotted in Figure 4. For purposes of comparison the exact velocity profile is also plotted in the same figure.

Note that if one chooses to define a new reduced velocity in terms of $(1 - u/U_f)$ rather than u/U_f, then the velocity distribution in terms of $\hat{\eta}$ becomes identical with that for a flat surface of finite length (*3*). This similarity however is only superficial, since the influence of the velocity component v on the u velocity profile is not considered.

Substituting Equation (15) in Equation (7*a*), integrating across the boundary layer, and combining with Equation (14) one obtains

$$U_f^2 \alpha \frac{d\delta}{dx} = \frac{g_c \tau_f}{\rho} \tag{18}$$

where

$$\alpha = \int_{\hat{\eta}=0}^{\hat{\eta}=1.0} f^2 d\hat{\eta} \tag{19}$$

The viscous shear stress at the surface is given by

$$\frac{g_c \tau_f}{\rho} = -\nu \left(\frac{\partial u}{\partial y}\right)_{y=0} =$$

$$-\frac{\nu U_f}{\delta} f'(O) = \beta \frac{\nu U_f}{\delta} \tag{20}$$

where

$$\beta = -f'(O) \tag{21}$$

and the prime here denotes differentiation with respect to $\hat{\eta}$. When one substitutes Equation (20) in Equation (18) and integrates from $\delta = 0$ at $x = 0$, there results the following equation for the boundary-layer thickness

$$\delta = \sqrt{\frac{2\beta}{\alpha}} \sqrt{\frac{\nu x}{U_f}} \tag{22}$$

The unknown coefficients α and β can now be evaluated from Equations (19) and (21), together with Equation (17). Thus $\alpha = 0.183$ and $\beta = 2.0$. The remaining boundary-layer parameters can be determined readily from Equations (7*a*), (8*a*), and Equations (10) to (13), together with Equations (17) and (22). The results are given in Table 2.

Rayleigh's Problem

There exists an approximate method, originally due to Rayleigh (*1*), for determining the boundary layer on a flat plate of infinite width. This involves the case of the flow near a flat plate of infinite width, which is suddenly accelerated from rest and moves in its own plane with a constant velocity U_o. The solution to this problem, which belongs to the class of parallel flows, is available (*3*). In this solution all the boundary-layer quantities are a function of the time from the start-up of motion. The transition from this problem to the case of a flat plate of infinite width, but finite in length, in steady motion in a fluid medium is made by substituting for t

$$t = \frac{x}{U_o} \tag{23}$$

where x is the distance from the leading edge of the plate. Rayleigh's solution may be applied to the moving continuous flat surfaces considered here by defining t as the time during which a solid particle of the surface was subjected to drag. Hence

$$t = \frac{x}{U_f} \quad (24)$$

where x is the distance from the slot (Figure 1).

The velocity profile is given by the relation

$$\frac{u}{U_f} = \text{erfc}\left(\frac{\eta}{2}\right) \quad (25)$$

Equation (25) is plotted in Figure 4.

The characteristic boundary-layer parameters determined on this basis are given in Table 2.

Discussion

For purposes of comparison the characteristic boundary-layer parameters obtained by the different methods of solution are summarized in Table 2.

The momentum thickness (which determines the drag on the moving continuous surface) obtained by the approximate integral solution is about 4% lower than that obtained by the exact solution of the boundary-layer equations. This proves again the ability of the approximate integral method to determine accurately the momentum thickness, despite the fact that the chosen velocity profile does not represent closely the true velocity profile, as shown in Figure 4. The displacement thickness (which determines the pumping action of the moving continuous surface) obtained by the approximate solution is about 13% lower than that obtained by the exact solution. For the flat plate of finite length the momentum and displacement thicknesses obtained by the approximate method (3) are about 4 and 1% higher, respectively, than the exact values.

The approximate method however cannot be used to establish the boundary-layer thickness accurately, as seen from Table 2. The same is true with the flat plate of finite length (3).

The results obtained by the Rayleigh solution are very approximate and are used here only for instructional purposes. This method of analysis however can be used with advantage in more complicated cases, as in Part III of this series. In this method the convection terms in the momentum equation are simplified by assuming that all convection takes place at the constant velocity of the solid surface. This implies that accelerated fluid is pumped away faster than is actually the case. Hence the boundary-layer thickness is underestimated, whereas the momentum thickness is overestimated. In this method the displacement and momentum thicknesses are equal.

The boundary-layer behavior on a moving continuous flat surface is quite different from that on a moving flat plate of finite length. The momentum thickness or drag on the continuous surface is about 34% higher than that on the surface of finite length. This indicates that the velocity profile near the continuous surface is steeper than that near the surface of finite length. Since the displacement thickness is largely determined by the velocity profile near the surface, it follows that the displacement thickness on the continuous surface should be smaller than that on the surface of finite length. This is actually the case, although the difference is only about 7%. The reason for the rather large displacement thickness on the continuous surface is to be found in the magnitude of the boundary-layer thickness. Clearly, the velocity near the edge of the boundary layer on the continuous surface does not change as fast as it does on the surface of finite length.

THE TURBULENT BOUNDARY LAYER ON A CONTINUOUS FLAT SURFACE

Approximate Solution

The only satisfactory method of determining the turbulent boundary layer on a given surface is based on the use of the momentum integral equation for the boundary layer. This equation is valid for both laminar and turbulent boundary layers and has already been derived (2). With suitable empirical relations for the velocity profile in the boundary layer and the shearing stress at the solid surface, the momentum integral equation yields the sought relations between the characteristic parameters of the boundary layer.

The momentum integral equation is given by Equation (14). The assumption will be made that the velocity profile is represented by the relation

$$\frac{u}{U_f} = 1 - \hat{\eta}^{1/7} \quad (26)$$

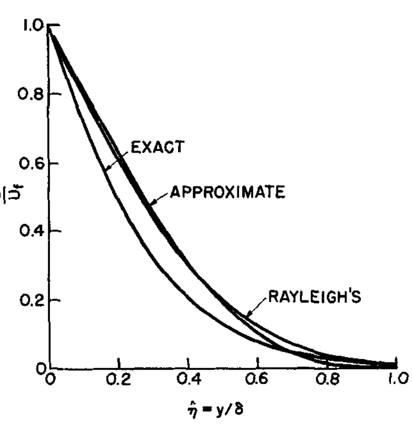

Fig. 4. Velocity distribution in the boundary layer along a continuous flat surface.

TABLE 2. LAMINAR BOUNDARY LAYER ON FLAT SURFACES, CHARACTERISTIC BOUNDARY-LAYER PARAMETERS

Continuous surfaces: Exact solution	Continuous surfaces: Integral solution	Continuous surfaces: Rayleigh's solution	Surfaces of finite length* (3): Exact solution
Boundary-layer thickness $\delta/\sqrt{\nu x/U_f}$			
6.37	4.68	3.64	4.91
Displacement thickness $\delta^*/\sqrt{\nu x/U_f}$			
1.616	1.40	1.128	1.729
Momentum thickness $\theta/\sqrt{\nu x/U_f}$			
0.888	0.856	1.128	0.664
Local coefficient of skin friction $C_f\sqrt{(N_{Re})_x}$			
0.888	0.856	1.128	0.664

* Solution for moving flat plate of finite length. For this case U_f corresponds to U_0.

Equation (26) is equivalent to the velocity profile represented by

$$\frac{u}{U_\bullet} = \hat{\eta}^{1/7}$$

which is valid for a moving flat plate of finite length at moderate Reynolds numbers. The assumption will also be made that the law of the shear stress at the solid surface is given by the relation

$$\frac{g_c \tau_f}{\rho U_f^2} = 0.0225\left(\frac{\nu}{U_f\delta}\right)^{1/4} \quad (27)$$

This is the same relation used for determining the turbulent boundary layer on a flat plate of finite length with zero pressure gradient along the plate.

The momentum thickness is determined readily from Equations (7a) and (26); hence

$$\theta = \frac{1}{U_f^2}\int_{y=0}^{y=\infty} u^2 dy = \int_{\hat{\eta}=0}^{\hat{\eta}=1.0} [1 - \hat{\eta}^{1/7}]^2 \,\delta d\hat{\eta} = \frac{1}{36}\delta \quad (28)$$

Substituting Equations (27) and (28) in Equation (14), rearranging, and integrating from the initial value $\delta = 0$ at $x = 0$ one gets the following equation for the boundary-layer thickness:

$$\delta = 1.01\, x \left(\frac{U_f x}{\nu}\right)^{-1/5} \quad (29)$$

Table 3. Turbulent Boundary Layer on Flat Surfaces Characteristic Boundary-Layer Parameters

Continuous surfaces Integral solution	Surfaces of finite length* (3) Integral solution
Boundary-layer thickness $(\delta/x)(N_{Re})_x^{1/5}$	
1.01	0.37
Displacement thickness $(\delta^*/x)(N_{Re})_x^{1/5}$	
0.126	0.046
Momentum thickness $(\theta/x)(N_{Re})_x^{1/5}$	
0.028	0.036
Local coefficient of skin friction $C_f'\ (N_{Re})_x^{1/5}$	
0.0449	0.0576

* Solution for moving flat plate of finite length. For this case U_f corresponds to U_o.

The remaining characteristic parameters are obtained from Equation (8*a*) and Equations (10) to (13), together with Equations (26) and (28). The results are given in Table 3.

Discussion

The characteristic parameters for the turbulent boundary layer are summarized in Table 3. For purposes of comparison the characteristic parameters for the turbulent boundary layer on a flat plate of finite length are also given.

The momentum thickness or drag on a moving continuous surface was found to be about 22% smaller than the drag on a moving flat plate of finite length. This result is surprising, in view of the conclusion reached for these two cases in the laminar-flow calculations. The calculations for the turbulent boundary layer are based on the use of the momentum integral equation. Two main assumptions were made in these calculations with regard to the velocity profile in the boundary layer and the law of shear stress at the solid surface. The momentum thickness however is not sensitive to small differences in the velocity distribution. Thus if the 1/7 exponent in the velocity profile given by Equation (26) is replaced by a 1/9 or 1/5 exponent, the constant in the momentum thickness expression (Table 3) will become 0.0258 and 0.0312, respectively. It is not likely that the exponent in the velocity profile will be greater than 1/5. Owing to lack of experimental evidence the law of shear stress at the surface of the moving continuous solid surface [Equation (27)] was taken to be the same as that on a *solid surface of finite length*. However the shear-stress law is a local function that should be applicable equally well to both cases. Furthermore the same shear-stress law is applicable to flow with pressure gradient in circular pipes. The two flows considered in Table 3 have a greater similarity than the flow over a flat plate of finite length as compared with the flow through a circular pipe. The conclusion therefore is that the results for the turbulent boundary layer on a moving continuous surface are reliable, at least with regard to the momentum thickness. Confirmation of this by experimentation would be desirable.

The results for the flat plate of finite length presented in Table 3 are valid in the range of $5 \times 10^5 < (N_{Re})_x < 10^7$. At higher Reynolds numbers the equations underestimate the actual drag. The same limits of validity may be applied by analogy to moving continuous flat surfaces.

The results presented in Table 3 are applicable directly if the boundary layer is turbulent from the leading edge onwards. In most cases the boundary layer is laminar to begin with and changes to turbulent farther downstream. The point of transition will depend to some extent on a number of variables, such as surface roughness and flutter of the moving surface. In the absence of any experimental evidence the critical Reynolds number for the transition point may be taken as

$$(N_{Re})_x \cong 5 \times 10^5$$

When the laminar portion of the boundary layer is a substantial fraction of the total exposed length of the surface, the procedure recommended by Prandtl (3) for calculating the total drag may be used.

SUMMARY

In this paper (Part II of this series) the laminar and turbulent boundary layers on a continuous flat surface were investigated by both exact and approximate methods. Relations were developed for the boundary-layer thickness, displacement thickness, momentum thickness, and skin friction, for both laminar and turbulent boundary layers; these relations are summarized in Tables 2 and 3.

Significant differences were found between the boundary-layer behavior on a continuous flat surface and a flat surface of finite length. The drag on the continuous flat surface was found to be higher than that on an equivalent flat surface of finite length for the laminar boundary layer but lower for the turbulent boundary layer.

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NOTATION

A = $2bL$ = exposed area of flat surface, sq. ft.
b = width of flat surface, ft.
C_f' = coefficient for local skin friction, dimensionless
C_f = coefficient for total skin friction, dimensionless
D = drag, lb. force
f = stream function in terms of parameter η or $\hat{\eta}$, dimensionless
g_c = conversion factor, 32.2 (lb.)(ft.)/(lb. force) (sec.)^2
L = exposed length of flat surface, ft.
q = volumetric flux, cu. ft./sec.
$(N_{Re})_x$ = $U_f x/\nu$ = local Reynolds number, dimensionless
t = time, sec.
U_f = velocity of continuous surface, ft./sec.
U_o = velocity of flat surface of finite length, ft./sec.
u = fluid-velocity component in the x direction, ft./sec.
v = fluid-velocity component in the y direction, ft./sec.
x = coordinate of Cartesian system of axes, ft.
y = coordinate of Cartesian system of axes, ft.

Greek Letters

α = constant, dimensionless
β = constant, dimensionless
δ = boundary-layer thickness, ft.
δ^* = displacement thickness, ft.
δ^*_L = displacement thickness at distance L, ft.
η = $y\sqrt{U_f/\nu x}$ = dimensionless parameter
$\hat{\eta}$ = y/δ = dimensionless parameter
θ = momentum thickness, ft.
θ_L = momentum thickness at distance L, ft.
μ = fluid viscosity, (lb.)/(ft.)(sec.)
ν = μ/ρ, kinematic fluid viscosity, sq. ft./sec.
ρ = fluid density, lb./cu. ft.
τ_f = shear stress on continuous flat surface, lb. force/sq. ft.
ψ = stream function, sq. ft./sec.

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